[8] Peters, C. E., "Turbulent Mixing and Burning of Coaxial Streams Inside a Duct of Arbitrary Shape," TR 68-270, Jan. 1969, Arnold Engineering Development Center, Tenn.

[9] Paulk, R. A., "Experimental Investigation of Free Turbulent Mixing of Nearly Constant Density Coaxial Streams," M.S. thesis, March 1969, University of Tennessee, Knoxville, Tenn.

[10] Peters, C. E., Chriss, D. E., and Paulk, R. A., "Turbulent Transport Properties in Subsonic Coaxial Free Mixing Systems," AIAA Paper 69-681, San Francisco, Calif., 1969.

[11] Chriss, D. E., "Experimental Study of the Turbulent Mixing of Subsonic Axisymmetric Gas Streams," TR 68-133, Aug. 1968, Arnold Engineering Development Center, Tenn.

Self-Correcting Initial Value Formulations in Nonlinear Structural Mechanics

JAMES A. STRICKLIN* AND WALTER E. HAISLER†
Texas A&M University, College Station, Texas
AND
WALTER A. VON RIESEMANN‡
Sandia Corporation, Albuquerque, N. Mex.

Introduction

THE purpose of this Note is to explore the initial value formulation for nonlinear static structural mechanics problems. It is assumed that the nonlinearities are due to small plastic strains and/or large deflections.

Probably the most widely used methods for nonlinear structural analysis by the finite element approach are the incremental procedure of Turner, Dill, Martin and Melosh[1] for large deflections and the tangent stiffness method of Pope[2] for plastic strains. In either case, the incremental procedure requires that new stiffness matrices be computed and "inverted" at each load increment and, consequently, requires the expenditure of large amounts of computer time. Until recently, however, the incremental and Newton-Raphson approaches were the only methods capable of solving for highly nonlinear behavior. In this Note, an alternative approach, a self-correcting initial value formulation, is presented which may be used to solve for highly nonlinear behavior. The method is demonstrated by solving for the geometrically nonlinear behavior of a single degree-of-freedom system and for the deflection of a spherical cap under a point load at the apex.

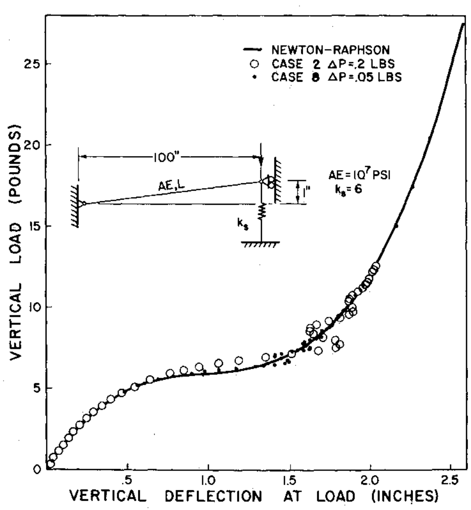

Fig. 1 Load deflection curve for truss.

Received March 17, 1971; revision received May 21, 1971. Research supported by Sandia Contract 82-5617 and NASA Grant NGL-44-011-044 and by the United States Atomic Energy Commission.

Index category: Structural Static Analysis.

* Professor, Department of Aerospace Engineering. Member AIAA.

† Assistant Professor, Department of Aerospace Engineering. Associate Member AIAA.

‡ Member, Technical Staff. Member AIAA.

Initial Value Formulation

The analysis of nonlinear problems by either the finite element method or the finite difference method gives rise to the set of equations:

$$[K]\{q\} = P\{\bar{P}\} + \{Q\} \tag{1}$$

where $[K]$ = linear stiffness matrix, $\{q\}$ = generalized displacements, $P\{\bar{P}\}$ = generalized forces due to applied external loads, P = convenient normalizing factor, $\{Q\}$ = pseudo generalized forces due to nonlinearities and dependent on the displacements.

Taking the derivative of Eq. (1) with respect to the scalar P yields

$$[K]\{\dot{q}\} = \{\bar{P}\} + \{\dot{Q}\} \tag{2}$$

where a dot indicates differentiation with respect to P.

Equation (2) is one form which has been solved. By using chain rule differentiation on $\dot{Q}$, another form of Eq. (2) is obtained.

$$\{\dot{Q}\} = [K^*]\{\dot{q}\} \tag{3}$$

where the elements of K^* are given by

$$k_{ij}^* = \partial Q_i/\partial q_j$$

Substituting Eq. (3) into Eq. (2) yields the equation

$$([K] + [K^*])\{\dot{q}\} = \{\bar{P}\} \tag{4}$$

The solution of Eqs. (2) and (4) which tends to drift away from the correct solution may be corrected by adding, to either Eq. (2) or (4), the unbalance of force f given by

$$\{f\} = -[K]\{q\} + P\{\bar{P}\} + \{Q\} \tag{5}$$

Multiplying Eq. (5) by an arbitrary factor Z and adding the result to the righthand side of Eqs. (2) and (4) yields:

$$[K](\{\dot{q}\} + Z\{q\}) = (1 + ZP)\{\bar{P}\} + Z(\{Q\} + \{\dot{Q}\}1/Z) \tag{6}$$

$$([K] + [K^*])\{\dot{q}\} + Z[K]\{q\} = (1 + ZP)\{\bar{P}\} + Z\{Q\} \tag{7}$$

Equations (6) and (7) are self-correcting first-order differential equations and to the author's knowledge have not been presented previously. If Z = 1.0 and an Euler forward

Table 1 Single degree-of-freedom studies[a]

Case	$Z(P)^{1/2}$	ΔP	Maximum load, lb	Comment
1	224	0.2	79	Numerically unstable
2	112	0.2	264	Numerically unstable
3	56	0.2	870	Numerically unstable
4	56	0.2	>1160	Stable
5	632	0.1	142	Numerically unstable
6	316	0.1	>600	Stable
7	1789	0.05	>300	Stable
8	894	0.05	>300	Stable

[a] $C = Z^{0.2}/2$ except for Case 4 where $C = 0.2\,(\Delta P)Z$.

difference is used for $\dot{q}$ in Eq. (7), the corrective procedure given in Refs. 3 and 4 is obtained.

A second derivative form is obtained by differentiating Eq. 2 with respect to P and adding the unbalance in force and its first derivative to the righthand side

$$[K]\{\ddot{q}\} = \{\ddot{Q}\} + Z\{f\} + C\{\dot{f}\} \tag{8}$$

where C is an arbitrary factor. Using Eq. (5), Eq. (8) becomes:

$$[K](\{\ddot{q}\} + C\{\dot{q}\} + Z\{q\}) = (C + ZP)\{\bar{P}\} + Z[\{Q\} + C/Z\{\dot{Q}\} + (1/Z)\{\ddot{Q}\}] \tag{9}$$

The lefthand side of Eq. (9) is the equation for simple harmonic motion with a damping coefficient C and an undamped natural frequency of $(Z)^{1/2}$rad/lb, and thus should provide insight into selecting values for C and Z. The righthand side may be interpreted as a forcing function. In this investigation the $\ddot{Q}$ term has been neglected.

Solution Procedure

Equation (9) is solved by using a four point backwards difference (Houbolt) for $\ddot{q}$ and a three point backwards difference for $\dot{q}$. Starting values for the solution procedure are obtained by successive approximations. The recursion relation involves a single matrix multiplication or solution and the addition of four vectors, and due to this simple relation, the computer time used per load increment is quite small. However, the biggest saver of computer time is the fact that the stiffness matrix is "inverted" only once for the solution.

Sample Problems

The first problem is a single-degree-of-freedom system shown in Fig. 1. The different cases are tabulated in Table 1. The following points should be noted with regard to Table 1: 1) Changing the value for C in going from case 3 to case 4 increases the stability. 2) A greater than sign (>) simply indicates that the case was terminated due to some reason other than numerical instability. 3) The stability is very much dependent on the load increment and the value selected for Z. A reduction in either variable enhances stability. 4) The solution by successive approximations diverges at 22 lbs. Thus, all the cases represent highly nonlinear behavior.

Figure 1 presents the solutions for cases 2 and 8 for low values of the load and compares the results with a Newton-Raphson solution. The oscillatory nature of the solution is noted in both cases with the higher-frequency oscillations occurring for the larger values of Z. In all cases, however, the oscillations quickly subside and the solution returns to the true solution.

One case was run without the spring shown in Fig. 1 which produces a snap buckling problem. It was found that Eq. (9) yielded a solution in the prebuckled state and then went to the postbuckled state. After a few oscillations, the solution converged to the postbuckled solution. This has interesting implications with regard to postbuckling analyses and will be reported elsewhere.

The second problem chosen for study was a shallow spherical cap under a point load at the apex (Fig. 2). The matrix displacement method was used for the analysis. This particular problem was selected because it exhibits highly nonlinear behavior. For example, the method of successive approximation may be used only to loads below 15 lb. The results for the deflection at the apex vs load are presented in Fig. 2 with the definitions of the different cases being given in Table 2.

Table 2 Shallow cap studies

Case	$Z(P)^{1/2}$	ΔP	C	Maximum load, lb	Comments
9	6.7	0.2	$0.2(\Delta P)Z$	>275	Stable
10	126	0.1	$Z^{0.2}/2$	50	Numerically unstable
11	447	0.05	$Z^{0.2}/2$	>150	Stable

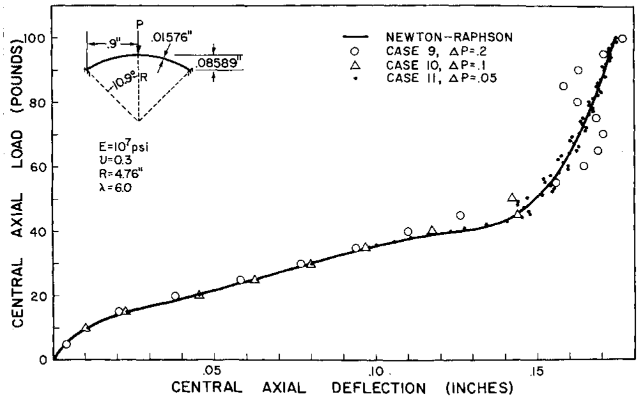

Fig. 2 Load deflection curve for shallow cap.

The results presented in Table 2 again show that stability is strongly influenced by the load increment and the value selected for Z. However, surprising is the accuracy that may be obtained using a load increment of 0.2 lb and a $Z(P)^{1/2}$ of 6.7. This corresponds to a value of Z equal to 1.0 at 45 lb of load. Despite the low value of Z, the solution still oscillates about the true solution. In all cases, the oscillations subside after a few cycles.

No attempt was made to compute the stresses in these example problems. However, no difficulty is expected as Eq. (9) shows that all degrees of freedom have the same natural frequency and damping coefficient. Thus, at loads where oscillations occur, the stresses should also oscillate about the true solution.

Conclusions and Recommendations

Enough experience has not been obtained with the self-correcting initial value formulation for a complete evaluation, and in its present form it may not be highly accurate. However, it is believed that the results presented definitely show the feasibility of the method and justify pursuing the research further; in particular, the method should be appropriate for design purposes where long computer runs are not desirable. In this regard, the following lines of pursuit are recommended: 1) Use of a variable load increment with a smaller increment being used at the larger loads. 2) Pursue other methods of solution. The very simple form of the lefthand side of Eq. (9) may lead to a solution procedure which is both accurate and stable. 3) More research is needed on the proper selection of Z and C. 4) For plastic nonlinearities, the oscillatory behavior must be removed.

References

[1] Turner, M. J., Dill, E. H., Martin, H. C., and Melosh, R. J., "Large Deflections of Structures Subjected to Heating and External Loads," *Journal of the Aerospace Sciences*, Vol. 27, No. 2, Feb. 1960, pp. 99–106.

[2] Pope, G. G., "A Discrete Element Method for Analysis of Plane Elasto-Plastic Stress Problems," *The Aeronautical Quarterly*, Vol. 17, 1966, pp. 83–104.

[3] Stricklin, J. A., Haisler, W. E., and Von Riesemann, W. A., "Geometrically Nonlinear Structural Analysis by the Direct Stiffness Method," *Journal of the Engineering Mechanics Division* submitted for publication, Aug. 1970.

[4] Hofmeister, L. D., Greenbaum, G. A., and Evensen, D. A., "Large Strain, Elastic-Plastic Finite Element Analysis," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1248–1254.